Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

BOUNDARY PROBLEMS AND GREEN'S FUNCTIONS FOR LINEAR DIFFERENTIAL AND DIFFERENCE EQUATIONS.

By Maxime Bôcher.

In Part I of the present paper I have established some simple results of a general character concerning linear boundary problems in one dimension. These results were established by Mason* for the equation of the second order by a method which admits of easy generalization. The proof here given of the main result (Theorem 3) is, however, of an even more elementary character than that given by Mason.

Part II is devoted to Green's Functions in one dimension. This conception was introduced by Burkhardt† for the special differential equation $y'' = 0$ (Laplace's Equation in one dimension), and extended by the present writer to the general linear differential equation of the nth order.‡ No proofs were given in this paper since it was thought, apparently erroneously, that the methods used would be sufficiently obvious. Subsequent writers, notably Westfall,§ established some of these results together with some similar ones for certain other boundary conditions, but, owing to their failure to appreciate the statement clearly made in my note just referred to that the Green's function will in general satisfy *different* boundary conditions when regarded as a function of one of its arguments from those which it satisfies when regarded as a function of the other, they failed to consider the general case. This general case, which includes as a special case the boundary conditions of my note, was first explicitly considered, though without proofs, by Birkhoff.|| Still more recently an exhaustive treatment of the subject for systems of linear differential equations of the first order has been given by Bounitzky,** who also applies his results at length to the equation of the nth order. In this paper the earlier literature is fully cited with the exception of Birkhoff's paper, which really contains the whole theory in brief so far as the single equation of the nth order goes. In Part II of the present paper I publish for the first time, along with a small

---

* Trans. Amer. Math. Soc., vol. 7 (1906), p. 339.

† Bull. de la Soc. Math. de France, vol. 22 (1894), p. 71.

‡ Bull. Amer. Math. Soc., second series, vol. 7 (1901), p. 297.

§ Dissertation, Göttingen (1905).

|| Trans. Amer. Math. Soc., vol. 9 (1908), p. 377.

** Liouville's Journal, 6th series, vol. 5 (1909), p. 65.

6

amount of additional matter, the method of treatment I had in mind ten years ago, modified only to the slight extent necessary to make it apply to the general linear boundary conditions.

Finally in Part III it is shown briefly how the results of Parts I and II can be carried over to the case of linear *difference* equations. The details are, in the main, fairly obvious, and are therefore to a considerable extent suppressed.

I. Boundary Problems for Ordinary Linear Differential Equations.

1. Let us consider the differential equation

$$\frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \cdots + p_n u = p, \tag{1}$$

where the p's are continuous but not necessarily real functions of the real variable x when $a \leqq x \leqq b$, together with certain boundary conditions which we will indicate as follows:

If φ is a function of x which at the points a and b has derivatives of the first $n-1$ orders, we write

$$\begin{aligned} A_i(\varphi) &\equiv \alpha_i \varphi(a) + \alpha_i' \varphi'(a) + \cdots + \alpha_i^{[n-1]} \varphi^{[n-1]}(a), \\ B_i(\varphi) &\equiv \beta_i \varphi(b) + \beta_i' \varphi'(b) + \cdots + \beta_i^{[n-1]} \varphi^{[n-1]}(b), \end{aligned} \qquad (i = 1, 2, \cdots n) \tag{2}$$

where the α's and β's are given constants. If now we let

$$W_i(\varphi) \equiv A_i(\varphi) + B_i(\varphi), \tag{3}$$

we may write our boundary conditions as follows, the γ's being given constants:

$$W_i(u) = \gamma_i \qquad (i = 1, 2, \cdots n). \tag{4}$$

The problem of determining all solutions of (1) which satisfy conditions (4) we speak of for brevity as the problem (1), (4); and this problem we call *homogeneous* only when $p \equiv 0$ and all the constants γ_i vanish. Two cases of some importance may be designated as *semi-homogeneous* problems, namely that in which $p \equiv 0$ but at least one γ_i is not zero, and also that in which all the γ_i's vanish but p is not identically zero.

The homogeneous equation

$$\frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \cdots + p_n u = 0 \tag{5}$$

is called the *reduced equation* when considered in connection with the non-homogeneous equation (1), and the homogeneous boundary conditions

$$W_i(u) = 0 \qquad (i = 1, 2, \cdots n) \tag{6}$$

we will call the *reduced boundary conditions* when we consider them in connection with (4). The homogeneous system (5), (6) we may similarly speak of as the *reduced system.*

DEFINITION. *The homogeneous system* (5), (6) *is said to be incompatible if* (5) *has no solution except zero which satisfies* (6). *It is said to have k-fold compatibility if* (5) *has k and only k linearly independent solutions which satisfy* (6).

If (5), (6) has k-fold compatibility and if $y_1, \cdots y_k$ are linearly independent solutions of (5), (6), it is clear that the general solution is

$$c_1y_1 + c_2y_2 + \cdots + c_ky_k$$

where $c_1, \cdots c_k$ are arbitrary constants.

THEOREM 1. *If $y_1, \cdots y_n$ is a fundamental system of* (5), *a necessary and sufficient condition that the system* (5), (6) *be compatible is that the determinant*

$$D = \begin{vmatrix} W_1(y_1) & \cdots & W_1(y_n) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ W_n(y_1) & \cdots & W_n(y_n) \end{vmatrix}$$

*vanish.** *A necessary and sufficient condition that* (5), (6) *have k-fold compatibility is that D be of rank $n - k$.*

For if we substitute the general solution of (5)

$$u = c_1y_1 + \cdots + c_ny_n$$

into (6), we obtain a system of homogeneous linear equations for determining the c's whose determinant is D.

We shall naturally call the conditions (6) linearly dependent when and only when the n sets of $2n$ constants each

$$\alpha_i, \alpha_i', \cdots \alpha_i^{[n-1]}, \beta_i, \beta_i', \cdots \beta_i^{[n-1]} \qquad (i = 1, 2, \cdots n)$$

are linearly dependent. If this is the case, some of the conditions (6) are consequences of the rest, and the system (6) is equivalent to a system of the same form containing less than n equations. Conversely, a system of the form (6) but containing less than n equations is obviously equivalent to a linearly dependent system containing n equations. On the other hand, if the system (6) is linearly dependent, $D = 0$. Hence

THEOREM 2. *There always exists a solution of* (5) *not identically zero which satisfies fewer than n conditions of the form* (6).

---

* Cf. Birkhoff, loc. cit.

On the other hand, we readily see from Theorem 1 that if the conditions (6) are linearly independent, the system (5), (6) is *in general* incompatible, since in the space of $2n^2$ dimensions determined by the coordinates $\alpha_i, \cdots \alpha_i^{[n-1]}, \beta_i, \cdots \beta_i^{[n-1]}$ the equation $D = 0$ determines an algebraic manifold of only $2n^2 - 1$ dimensions.*

THEOREM 3. *A necessary and sufficient condition that the system* (1), (4) *have one and only one solution is that the reduced system* (5), (6) *be incompatible.*

A considerable part of this theorem follows at once from the obvious fact that *if* (1), (4) *admits of a solution at all, its general solution may be obtained by adding to any particular solution of* (1), (4) *the general solution of the reduced system* (5), (6). If then (5), (6) is compatible, (1), (4) has either no solution or an infinite number of solutions; while if (5), (6) is incompatible (1), (4) cannot have more than one solution. To prove our theorem completely, we need, then, merely to show that *if* (5), (6) *is incompatible,* (1), (4) *admits a solution.*

To prove this we begin with the semi-homogeneous case $p \equiv 0$. We see then from Theorem 2 that there exist n solutions $y_1, \cdots y_n$ of (1), no one of which vanishes identically, and such that y_i satisfies all of the reduced conditions (6) *except the ith.* Since the system (5), (6) is to be incompatible, none of the constants $W_i(y_i)$ are zero. Consequently the function

$$\frac{\gamma_1 y_1}{W_1(y_1)} + \cdots + \frac{\gamma_n y_n}{W_n(y_n)}$$

is a solution of (1) which satisfies all the conditions (4).

Turning now to the general case, let us denote by u any particular solution of (1), and by y the solution of (5) which satisfies the conditions

$$W_i(y) = \gamma_i - W_i(u) \qquad (i = 1, 2, \cdots n).$$

The existence of this function y is established by the part of the theorem we have just proved. The function $u + y$ is now clearly a solution of (1), (4). Thus our theorem is proved.

II. Green's Functions.

2. Definition and Condition for Existence.—We may arrive at the conception of the Green's Function for the system (5), (6) by attempting, in case the system is incompatible, to find a function not indentically zero which comes as near as possible to being a solution of the system, the only failure lying in a discontinuity in its $(n-1)$th derivative at a single point ξ. Since it is only when such a function exists for *every* value of ξ between a and b that it proves to be of any importance, we lay down the definition as follows:

---

* To clinch this argument it is necessary to know that D does not vanish identically, i. e., for all possible conditions (6). This can be shown by exhibiting any particular system (6) for which $D \neq 0$; and such a system is that in which $\alpha_1 = \alpha_2' = \alpha_3'' = \ldots = \alpha_n^{[n-1]} = 1$ while all the other α's and all the β's are zero.

DEFINITION. *By a Green's Function* $G(x, \xi)$ *of the system* (5), (6), *where we assume the conditions* (6) *to be linearly independent, we understand a function of* (x, ξ) *defined when* $a \leqq x \leqq b$, $a < \xi < b$, *and which for every such value of* ξ *when regarded as a function of* x *alone has the following properties*:

1) *Throughout the interval* $a \leqq x \leqq b$ *it is continuous and has continuous derivatives of the first* $n-2$ *orders.*

2) *At every point of the interval* $a \leqq x \leqq b$ *except* $x = \xi$ *it satisfies* (5).

3) *It fulfills the boundary conditions* (6).

4) *At* ξ *it has a forward derivative of order* $n-1$, D_+, *and a backward derivative of order* $n-1$, D_-; *and*

$$D_+ - D_- = 1.$$

It will be noticed that in this definition we do not demand that the system (5), (6) be incompatible; and even if it is incompatible we do not as yet know that a Green's function will exist, or, if it does exist, whether it will be uniquely determined. We proceed to investigate these questions.

Let $y_1, \cdots y_n$ be a fundamental system of (5), and form with undetermined coefficients the two solutions

$$u_1(x) = c_1 y_1(x) + \cdots + c_n y_n(x),$$
$$u_2(x) = d_1 y_1(x) + \cdots + d_n y_n(x).$$

The most general function which satisfies 2) of our definition is

$$G(x, \xi) = \begin{cases} u_1(x) & a \leqq x \leqq \xi \\ u_2(x) & \xi \leqq x \leqq b. \end{cases} \tag{7}$$

In order that conditions 1) and 4) of our definition be satisfied by this function, it is necessary and sufficient that the c's and d's satisfy the following equations:

$$\begin{aligned} &d_1 y_1^{[i]}(\xi) + \cdots + d_n y_n^{[i]}(\xi) - c_1 y_1^{[i]}(\xi) - \cdots - c_n y_n^{[i]}(\xi) = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 0, 1, \cdots n-2) \\ &d_1 y_1^{[n-1]}(\xi) + \cdots + d_n y_n^{[n-1]} c(\xi) - c_1 y_1^{[n-1]}(\xi) - \cdots - c_n y_n^{[n-1]}(\xi) = 1. \end{aligned} \tag{8}$$

This may be regarded as a system of linear equations for determining the n differences $d_i - c_i$. The determinant of these equations, being the

Wronskian of the y's, is not zero since the y's are linearly independent. The equations (8) have therefore one and only one solution which we will denote by $z_1, \cdots z_n$. These quantities, like the c's and d's, are of course functions of ξ, and we write explicitly

$$d_i(\xi) - c_i(\xi) = z_i(\xi). \tag{9}$$

These functions z_i may be immediately written out each as the ratio of two determinants by Cramer's Rule. They are precisely the functions defined by Frobenius* as the functions *adjoint* to $y_1, \cdots y_n$. It is evident that these functions are continuous throughout the interval $a \leqq \xi \leqq b$. Apart from this fact we need for the moment only one property of $z_1, \cdots z_n$, namely that they are linearly independent.†

In order that the function G in (7) satisfy condition 3) of our definition, it is necessary and sufficient that the c's and d's satisfy the following equations:

$$c_1A_i(y_1) + \cdots + c_nA_i(y_n) + d_1B_i(y_1) + \cdots + d_nB_i(y_n) = 0 \qquad (i = 1, 2, \cdots n), \tag{10}$$

or, after the c's have been replaced by their values from (9)

$$d_1W_i(y_1) + \cdots + d_nW_i(y_n) = z_1A_i(y_1) + \cdots + z_nA_i(y_n) \qquad (i = 1, 2, \cdots n). \tag{11}$$

This is a system of linear equations for determining the d's concerning which we will establish the following

LEMMA. *A necessary and sufficient condition that equations* (11) *be consistent for all values of* ξ *such that* $a < \xi < b$ *is that their determinant* D *do not vanish.*

That this is a sufficient condition is obvious from Cramer's Rule. To prove it necessary we suppose $D = 0$, and have to show that equations (11) are inconsistent. From the vanishing of D we infer that there exist n constants $k_1, \cdots k_n$, not all zero, such that

$$k_1W_1(y_i) + \cdots + k_nW_n(y_i) = 0 \qquad (i = 1, 2, \cdots n). \tag{12}$$

By multiplying equations (11) respectively by $k_1, \cdots k_n$ and adding, we could, if these equations were consistent, infer that

---

* Crelle, vol. 77 (1874), p. 250.

† The simple proof of this fact which consists in noticing that the Wronskian of the z's cannot vanish since the product of this Wronskian by the Wronskian of the y's is 1, is not available for us unless we make sufficient assumptions concerning the coefficients of (5) to secure the existence of the first $n-1$ derivatives of the z's. Cf., however, an article by the author: Bull. Amer. Math. Soc., 2d Ser., vol. 8 (1901), p. 53. See particularly Theorem 14, p. 61.

$$0 = z_1 \sum_{j=1}^{n} k_j A_j(y_1) + \cdots + z_n \sum_{j=1}^{n} k_j A_j(y_n).$$

Since this is an identity in ξ, and the z's are linearly independent, it follows that

$$k_1 A_1(y_i) + \cdots + k_n A_n(y_i) = 0 \qquad (i = 1, 2, \cdots n). \tag{13}$$

Subtracting these equations from (12), we find

$$k_1 B_1(y_i) + \cdots + k_n B_n(y_i) = 0 \qquad (i = 1, 2, \cdots n). \tag{14}$$

If we use formula (2) for the A's and B's, (13) and (14) take the form

$$(k_1\alpha_1 + \cdots + k_n\alpha_n) y_i(a) + \cdots + (k_1\alpha_1^{[n-1]} + \cdots + k_n\alpha_n^{[n-1]}) y_i^{[n-1]}(a) = 0 \tag{15}$$

$$(k_1\beta_1 + \cdots + k_n\beta_n) y_i(b) + \cdots + (k_1\beta_1^{[n-1]} + \cdots + k_n\beta_n^{[n-1]}) y_i^{[n-1]}(b) = 0 \tag{16}$$

$$(i = 1, 2, \cdots n).$$

We may regard (15) as a sytem of linear equations for determining the n quantities which occur in it in parentheses. The determinant of this system is the Wronskian of $y_1, \cdots y_n$ taken at the point $x = a$, which, since the y's are linearly independent, is not zero. Consequently

$$k_1\alpha_1^{[j]} + \cdots + k_n\alpha_n^{[j]} = 0 \qquad (j = 0, 1, \cdots n - 1). \tag{17}$$

Similarly we infer from (16) that

$$k_1\beta_1^{[j]} + \cdots + k_n\beta_n^{[j]} = 0 \qquad (j = 0, 1, \cdots n - 1). \tag{18}$$

These relations (17), (18) mean, however, that the conditions (6) are linearly dependent, a possibility, we have explicitly ruled out in the hypothesis of our definition of a Green's Function. One lemma is thus proved, since the assumption that it is false has led to a contradiction.

If we now notice that the D in our lemma is the same as the D in Theorem 1, we have the result:

THEOREM 4. *A necessary and sufficient condition for the existence of a Green's Function of the system* (5), (6), *where the boundary conditions* (6) *are assumed to be linearly independent, is that the system be incompatible.*

Since, moreover, if this condition is fulfilled, the determinant D of equations (11) is not zero, only one determination of the d's, and therefore by (9) of the c's, is possible. Hence

COROLLARY. *If the system* (5), (6) *is incompatible, it has only one Green's Function.*

The Green's Function has been defined so far only when ξ is equal neither to a nor to b. We now lay down the further

Definition. *The formula* (7) *which gives the Green's Function when* $a < \xi < b$ *shall serve to define it when* $\xi = a$ *and when* $\xi = b$.

Since the c's and d's obtained from (11) and (9) are clearly continuous functions of ξ when $a \leqq \xi \leqq b$, and $y_1, \cdots y_n$ are continuous together with their first $n - 2$ derivatives when $a \leqq x \leqq b$, it follows from (7) that $G(x, \xi)$ and its first $n - 2$ derivatives with regard to x are continuous in each of the triangles $a \leqq x \leqq \xi \leqq b$ and $a \leqq \xi \leqq x \leqq b$. Moreover none of these functions has any discontinuity on the line $x = \xi$. Hence

Theorem 5. *If the Green's Function of the system* (5), (6) *exists, it is a continuous function of* (x, ξ) *when* $a \leqq x \leqq b$, $a \leqq \xi \leqq b$, *and the same is true of its first* $n - 2$ *derivatives with regard to* x.

3. The Fundamental Application of Green's Functions.—The general solution of the problem (1), (4) is clearly the sum of the general solutions of the semi-homogeneous problems (1), (6) and (5), (4). We have had occasion to consider the latter problem incidentally in §1. If the reduced system (5), (6) is incompatible, the former problem has, as we know, one and only one solution. We wish to prove in this section that this solution is given by the formula

$$u(x) = \int_a^b G(x, \xi) p(\xi) d\xi, \tag{19}$$

where G is the Green's Function of (5), (6). Referring to formula (7), we see that we can write

$$u(x) = \int_a^x [d_1(\xi) y_1(x) + \cdots + d_n(\xi) y_n(x)] p(\xi) d\xi \\ - \int_b^x [c_1(\xi) y_1(x) + \cdots + c_n(\xi) y_n(x)] p(\xi) d\xi.$$

If we differentiate and simplify by means of (8), we find:

$$u^{[i]}(x) = \int_a^x [d_1(\xi) y_1^{[i]}(x) + \cdots + d_n(\xi) y_n^{[i]}(x)] p(\xi) d\xi \\ - \int_b^x [c_1(\xi) y_1^{[i]}(x) + \cdots + c_n(\xi) y_n^{[i]}(x)] p(\xi) d\xi \quad (i = 1, \cdots n - 1),$$

$$u^{[n]}(x) = \int_a^x [d_1(\xi) y_1^{[n]}(x) + \cdots + d_n(\xi) y_n^{[n]}(x)] p(\xi) d\xi \\ - \int_b^x [c_1(\xi) y_1^{[n]}(x) + \cdots + c_n(\xi) y_n^{[n]}(x)] p(\xi) d\xi + p(x).$$

Substituting these values of u and its derivatives in (1), and remembering that the y's are solutions of (5), we see that the function u determined by (19) satisfies (1). We also find from the values of u and its derivatives just obtained that

$$\begin{aligned} A_i(u) &= \int_a^b [c_1(\xi)A_i(y_1) + \cdots + c_n(\xi)A_i(y_n)]p(\xi)d\xi \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, 2, \cdots n). \\ B_i(u) &= \int_a^b [d_1(\xi)B_i(y_1) + \cdots + d_n(\xi)B_i(y_n)]p(\xi)d\xi \end{aligned}$$

A reference to (10) shows that the sum of these two quantities is zero. Thus we have proved

THEOREM 6. *If the system* (5), (6) *has a Green's Function* $G(x, \xi)$, *the solution of the semi-homogeneous problem* (1), (6) *is given by* (19), *and the first* $n - 1$ *derivatives of* u *may be obtained by differentiating* (19) *under the integral sign.*

In the special case in which $\alpha_1 = \alpha_2' = \cdots = \alpha_n^{[n-1]} = 1$, while all the other α's and all the β's are zero, formula (19) gives the *principal solution* of (1) at the point a, that is the solution which with its first $n - 1$ derivatives vanishes at a. In this case the c's are all zero by (10), and hence, by (9), the d's are the functions adjoint to the y's. Hence formula (19) reduces to the standard formula (Schlesinger's Handbuch, vol. 1, p. 78, (6)), obtainable by the method of variation of constants or otherwise, for this principal solution.

4. Further Properties of Green's Functions.—The facts to which we now come are less fundamental than those we have so far considered but they are commonly given great prominence in the development of the theory, and they are in themselves not without interest. In order to obtain them we must impose additional restrictions on the coefficients of (5).

We assume that $p_i(x)$ *has continuous derivatives of the first* $n - i$ *orders.*

This is sufficient to insure the existence and continuity of the first n derivatives of the adjoint functions $z_1(\xi), \cdots z_n(\xi)$, and to cause them to satisfy the adjoint differential equation:*

$$(20) \qquad (-1)^n \frac{d^n z}{d\xi^n} + (-1)^{n-1} \frac{d^{n-1}(p_1 z)}{d\xi^{n-1}} + \cdots - \frac{d(p_{n-1}z)}{d\xi} + p_n z = 0.$$

From (11) and (9) we see that the c's and d's are linear homogeneous combinations with constant coefficients of the z's. Consequently the c's and d's are also solutions of (20). Turning to (7), we see then that when x is constant, G also satisfies (20) for all values of ξ in the interval $a \leqq \xi \leqq b$

---

* This may be seen as follows:

Even with the mere assumption of continuity on the functions p_i, it is known (cf. Bull. Amer. Math. Soc., 2d Ser., vol. 8 (1901), p. 63) that the multipliers (integrating factors) of (5) constitute a linear family of which $z_1, \ldots z_n$ forms a basis. With the additional restrictions concerning the derivatives of the p's, it is known from Lagrange's Identity that all solutions of (20) are multipliers of (5). These solutions form a linear family involving n linearly independent functions. Consequently the solutions of (20) are identical with the linear family $c_1z_1 + \ldots + c_nz_n$.

except $\xi = x$, and that even at this point it satisfies (20) if we agree to consider only forward or only backward derivatives.

In order to compare these forward and backward derivatives, that is the derivatives with regard to ξ of u_1 and u_2 at the point $\xi = x$, we turn to equations (8). Differentiate each of these equations with regard to ξ, remembering that the c's and d's are themselves functions of ξ, and subtract from each equation thus obtained the next following equation (8). We thus get the system of $n - 1$ equations:

$$(8') \quad \begin{aligned} & d_1'y_1^{[i]}(\xi) + \cdots + d_n'y_n^{[i]}(\xi) - c_1'y_1^{[i]}(\xi) - \cdots - c_n'y_n^{[i]}(\xi) = 0 \\ & \qquad\qquad\qquad\qquad\qquad\qquad (i=0, 1, \cdots n-3), \\ & d_1'y_1^{[n-2]}(\xi) + \cdots + d_n'y_n^{[n-2]}(\xi) - c_1'y_1^{[n-2]}(\xi) - \cdots - c_n'y_n^{[n-2]}(\xi) = -1. \end{aligned}$$

If here we differentiate each equation and subtract from it the next following equation (8′), we get a system of $n - 2$ equations (8″), etc. Finally we collect together the *first* equations of the sets (8), (8′), (8″), $\cdots$, and thus obtain a new system

$$(21) \quad \begin{aligned} & d_1^{[i]}(\xi)y_1(\xi) + \cdots + d_n^{[i]}(\xi)y_n(\xi) - c_1^{[i]}(\xi)y_1(\xi) - \cdots - c_n^{[i]}(\xi)y_n(\xi) = 0 \\ & \qquad\qquad\qquad\qquad\qquad\qquad (i = 0, 1, \cdots n - 2), \\ & d_1^{[n-1]}(\xi)y_1(\xi) + \cdots + d_n^{[n-1]}(\xi)y_n(\xi) \\ & \qquad - c_1^{[n-1]}(\xi)y_1(\xi) - \cdots - c_n^{[n-1]}(\xi)y_n(\xi) = (-1)^{n-1}. \end{aligned}$$

These equations show us that the first $n - 2$ forward derivatives of G with regard to ξ at the point $\xi = x$ have the same values respectively as the corresponding backward derivatives; while if we denote by Δ_+ and Δ_- the $(n - 1)$th forward and backward derivatives respectively at $\xi = x$,

$$\Delta_+ - \Delta_- = (-1)^n.$$

We thus see that, except for the boundary conditions, G when regarded as a function of ξ, x being a parameter satisfies precisely the conditions for the Green's function of equation (20), or if n is odd, for the negative of this Green's function. Let us then see whether we can find a system of boundary conditions of the form (6) satisfied by G for each (constant) value of x. The important point here is that the coefficients α, β must be independent of of x.

Let us write

$$\begin{aligned} A(\varphi) &= \alpha\varphi(a) + \cdots + \alpha^{[n-1]}\varphi^{[n-1]}(a), \\ B(\varphi) &= \beta\varphi(b) + \cdots + \beta^{[n-1]}\varphi^{[n-1]}(b), \\ W(\varphi) &= A(\varphi) + B(\varphi), \end{aligned}$$

where the α's and β's are undetermined constants. We wish to determine these constants in the most general way possible so that $G(x, \xi)$ regarded as a function of ξ shall satisfy the relation $W(G) = 0$. From (7) we have (x being regarded as a constant in u_1, u_2 as well as in G):

$$W(G) = A(u_2) + B(u_1) = y_1(x)[A(d_1) + B(c_1)] + \cdots + y_n(x)[A(d_n) + B(c_n)].$$

Consequently $W(G)$ vanishes for all values of x when and only when all the coefficients of $y_1, \cdots y_n$ in this expression are zero; that is when and only when the α's and β's satisfy the following equations:

$$(22) \quad \alpha d_i(a) + \cdots + \alpha^{[n-1]} d_i^{[n-1]}(a) + \beta c_i(b) + \cdots + \beta^{[n-1]} c_i^{\,n-1]}(b) = 0 \qquad (i = 1, 2, \cdots n).$$

Lemma. *The equations* (22) *are linearly independent.*

Suppose this were not the case. Then there would exist n constants $k_1, \cdots k_n$ not all zero and such that

$$\begin{aligned} k_1 d_1^{[j]}(a) + k_2 d_2^{[j]}(a) + \cdots + k_n d_n^{[j]}(a) &= 0 \\ k_1 c_1^{[j]}(a) + k_2 c_2^{[j]}(a) + \cdots + k_n c_n^{[j]}(a) &= 0. \end{aligned} \qquad (j = 0, 1, \cdots n - 1).$$

Consequently the functions

$$k_1 d_1(\xi) + \cdots + k_n d_n(\xi),$$
$$k_1 c_1(\xi) + \cdots + k_n c_n(\xi)$$

vanish, together with their first $n - 1$ derivatives, at the points a and b respectively; so that, being solutions of (20), they vanish identically. Accordingly their difference, which by (9) is

$$k_1 z_1(\xi) + \cdots + k_n z_n(\xi),$$

vanishes identically. But this is impossible since, as we noted in §2, $z_1, \cdots z_n$ are linearly independent.

The equations (22), being thus linearly independent, have n linearly independent solutions, which we denote by

$$\bar{\alpha}_i, \bar{\alpha}_i', \cdots \bar{\alpha}_i^{[n-1]}, \bar{\beta}_i, \bar{\beta}_i', \cdots \bar{\beta}_i^{[n-1]} \qquad (i = 1, 2, \cdots n),$$

upon which all other solutions are linearly dependent.

Definition. *The linearly independent boundary conditions*

$$(23) \quad \bar{\alpha}_i u(a) + \cdots + \bar{\alpha}_i^{[n-1]} u^{[n-1]}(a) + \bar{\beta}_i u(b) + \cdots + \bar{\beta}_i^{[n-1]} u^{[n-1]}(b) = 0 \qquad (i = 1, 2, \cdots n),$$

together with the equation (20) *form a homogeneous system which we call the* *system adjoint to the system* (5), (6). *In forming this system it is assumed that the system* (5), (6) *is incompatible.*[*]

The facts proved in this section may now be summed up in the following theorem:

THEOREM 7. *If the system* (5), (6) *has a Green's function* $G(x, \xi)$, *then* $(-1)^n G(x, \xi)$ *is the Green's function of the adjoint system* (20), (23).[†]

From this it follows that the system (20), (23) is also incompatible since it is only incompatible systems which have a Green's function. Moreover the system adjoint to (20), (23) has, by Theorem 7, $G(x, \xi)$ as its Green's function. The differential equation of this system is (5), since any homogeneous linear differential equation is the adjoint of its adjoint. Moreover the boundary conditions of the system adjoint to (20), (23) are easily seen to be conditions (6), since in the proof of Theorem 7 it was shown that there is essentially only one system of boundary conditions in the parameter which a Green's function satisfies. Hence

THEOREM 8. *If the system* (5), (6) *is incompatible, its adjoint is also incompatible, and the adjoint of this last named system is the original system* (5), (6).

The theory of Green's functions can be put into a more elegant form by considering, with Hilbert, not the differential *equation* (5) but the differential *expression*

$$(23) \qquad L(u) \equiv a_0 \frac{d^n u}{dx^n} + a_1 \frac{d^{n-1} u}{dx^{n-1}} + \cdots + a_n u,$$

where the a's are functions of x continuous throughout the interval $a \leqq x \leqq b$, a_i has continuous derivatives of the first $n - i$ orders, and a_0 does not vanish at any point of this interval.

DEFINITION. *By the Green's function of the system consisting of the expression* (23) *and the conditions* (6) *is understood the quotient of the Green's function of the system consisting of the equation* $L(u) = 0$ *and the conditions* (6) *by* $a_0(\xi)$.

The properties of Green's functions in this sense follow at once from the results already obtained. For instance, if we denote the Green's function in the new sense by $G(x, \xi)$, it is still true that the solution of the non-homogeneous equation

$$L(u) = p$$

---

[*] The term *adjoint* in this sense and the conception with this degree of generality was first used by Birkhoff, loc. cit. Indeed he defines the conception in a more general case since he does not restrict the system (5), (6) to be incompatible. Cf. also Bounitzki, loc. cit. In a special case the conception occurs in a paper of Liouville (Liouville's Journal, vol. 3 (1838), p. 561). That this conception plays an important part in the theory of Green's functions was clearly indicated by the present writer in 1901 (cf. loc. cit.).

[†] The factor $(-1)^n$ was omitted by an oversight in my paper of 1901.

which satisfies the homogeneous conditions (6) is, provided G exists, given by formula (19).

Again, the system adjoint to (23), (6) consists of the expression adjoint to $L(u)$ and of the boundary conditions which $G(x, \xi)$ satisfies when regarded as a function of ξ. These boundary conditions, however, are not the same as those satisfied by the Green's function of the *equation* $L(u) = 0$ together with (6). With this understanding we may say that the Green's function of (23), (6) is, when regarded as a function of ξ, precisely the Green's function of the adjoint system. It seems hardly necessary to multiply illustrations or to go into proofs here.

We close by noting that *the Green's function of the expresssion* (23) *together with the conditions* (6) *is a covariant both with regard to change of independent and of dependent variable of the system consisting of the equation* $L(u) = 0$ *together with the conditions* (6). That is, if a change of independent variable

$$t = f(x), \quad \tau = f(\xi),$$

where $f(x)$ has continuous derivatives of the first n orders, and $f'(x)$ does not vanish in the interval $a \leqq x \leqq b$, is made both in (23) and in (6), the Green's function of the transformed system is equal to the Green's function of the original system multiplied by a power (the (-1)th) of $f'(\xi)$; and on the other hand a change of independent variable

$$u = \psi \cdot \eta,$$

where ψ is a given non-vanishing function of x which in the interval $a \leqq x \leqq b$ is continuous and has continuous derivatives of the first n orders, carries over the system (23), (6) into a system whose Green's function is equal to the product of the original Green's function by a power (the (-1)th) of $\psi(x)$ And finally, if the expression (23) is multiplied by a non-vanishing function $\varphi(x)$, which together with its first n derivatives is continuous in the interval $a \leqq x \leqq b$, the Green's function of the new system is equal to the original Green's function multiplied by a power (the (-1)th) of $\varphi(\xi)$.

The truth of these statements follows readily from the definition of Green's functions. As in § 2, we need not demand the existence of derivatives of the coefficients a_i.

III. The Linear Difference Equation.

5. Linear Boundary Problems.—We consider here the difference equation

$$(24) \qquad p_0(x)u(x+n) + p_1(x)u(x+n-1) + \cdots + p_n(x)u(x) = p(x)$$
$$(x = a, a+1, \cdots b-n),$$

where a and b, and consequently x, may without loss of generality be regarded as integers. We will assume that $p_0(x)$ and $p_n(x)$ do not vanish for any value of x from a to $b-n$ inclusive; and that $b-a \geqq n$.

Along with (24) we consider certain boundary conditions which we denote as follows:

If $\varphi(x)$ is a function of the integer x defined when $a \leqq x \leqq b$, we write

$$(25)\quad \begin{aligned} A_i(\varphi) &= \alpha_{n-1}^{[i]}\varphi(a+n-1) + \alpha_{n-2}^{[i]}\varphi(a+n-2) + \cdots + \alpha_0^{[i]}\varphi(a) \\ B_i(\varphi) &= \beta_{n-1}^{[i]}\varphi(b-n+1) + \beta_{n-2}^{[i]}\varphi(b-n+2) + \cdots + \beta_0^{[i]}\varphi(b) \\ &\qquad\qquad\qquad (i = 1, 2, \cdots n), \end{aligned}$$

where the α's and β's are constants, i. e., independent of x. If, now, we let

$$(26)\qquad W_i(\varphi) = A_i(\varphi) + B_i(\varphi),$$

we may write our boundary conditions as follows, the γ's being constants:

$$(27)\qquad W_i(u) = \gamma_i \qquad (i = 1, 2, \cdots n)$$

The system (24), (27) is in general non-homogeneous, and we distinguish between semi-homogeneous and homogeneous systems precisely as in the analogous case of §1. The *reduced system* is

$$(28)\quad p_0(x)u(x+n) + p_1(x)u(x+n-1) + \cdots + p_n(x)u(x) = 0 \qquad (x = a, a+1, \cdots b-n),$$

$$(29)\qquad W_i(u) = 0 \qquad (i = 1, 2, \cdots n).$$

We define compatibility, k-fold compatibility, for the homogeneous system (28), (29) precisely as in §1, and we see at once that Theorems 1, 2 hold here also if (5), (6) be replaced by (28), (29). We also see, as in §1, that if the conditions (29) are linearly independent, the system (28), (29) is *in general* incompatible, the special conditions (29) which show that the determinant D does not always vanish being

$$u(a) = u(a+1) = \cdots = u(a+n-1) = 0.$$

Finally the fundamental theorem here is

THEOREM 9. *A necessary and sufficient condition that the system* (24), (27) *have one and only one solution is that the reduced system* (28), (29) *be incompatible.*

The proof of this theorem is identical with the proof of Theorem 3 in §1.

6. Green's Functions for Linear Difference Equations.—Here too the conception of a Green's Function may be regarded as having its rise in the attempt to form a function not identically zero which comes as near as possible to satisfying the system (28), (29) when this system is incompatible.

DEFINITION. *By a Green's function* $G(x, \xi)$ *of the system consisting of the expression in* (28), *and conditions* (29), *where we assume that the conditions* (29) *are linearly independent, we understand a function of the integral arguments* (x, ξ) *defined when* $a \leqq x \leqq b$, $a \leqq \xi \leqq b - n$, *and such that when* ξ *is fixed,* G *regarded as a function of* x *satisfies the boundary conditions* (29); *and, except for the one value* $x = \xi$, *the equation* (28); *and is such that when substituted in the first member of* (28), *this first member takes the value* 1 *when* $x = \xi$.

In order to see under what circumstances such a function exists, we consider a system $y_1, \cdots y_n$ of linearly independent solutions of (28), and form from them with undetermined coefficients the further solutions

$$
\begin{aligned}
u_1(x) &= c_1 y_1(x) + \cdots + c_n y_n(x), \\
u_2(x) &= d_1 y_1(x) + \cdots + d_n y_n(x),
\end{aligned}
\tag{30}
$$

where the c's and d's are ultimately to be functions of ξ; a fact which will be indicated when necessary.

The most general function which satisfies (28) except when x has one of the values $\xi, \xi - 1, \cdots \xi - n + 1$ lying in the interval $a \leqq x \leqq b - n$ is then

$$
u(x) = \begin{cases} u_1(x) & x = a, a+1, \cdots \xi, \\ u_2(x) & x = \xi + 1, \xi + 2, \cdots b. \end{cases}
\tag{31}
$$

In order that $u(x)$ satisfy (28) also at such of the points $x = \xi - 1, \cdots \xi - n + 1$ as satisfy the inequality $a \leqq x$ it is evidently necessary* and sufficient that

$$
u_2(x) = u_1(x) \qquad x = \xi + 1, \xi + 2, \cdots \xi + n - 1.
\tag{32}
$$

In order that the condition demanded in the last clause of our definition be fulfilled it is then necessary and sufficient that

$$
p_0(\xi)[u_2(\xi + n) - u_1(\xi + n)] = 1.
\tag{33}
$$

Substituting in (32), (33) the value of u_1 and u_2 from (30), and setting

$$
z_i(\xi) = p_0(\xi)(d_i(\xi) - c_i(\xi)),
\tag{34}
$$

we find as an equivalent form for conditions (32), (33)

$$
\begin{aligned}
z_1(\xi) y_1(\xi + i) + \cdots + z_n(\xi) y_n(\xi + i) &= 0 \qquad (i = 1, 2, \cdots n-1) \\
z_1(\xi) y_1(\xi + n) + \cdots + z_n(\xi) y_n(\xi + n) &= 1.
\end{aligned}
\tag{35}
$$

---

* Strictly speaking this is necessary only when $a \leqq \xi - n + 1$, so that *all* the points in question lie in the interval ab. If some of the points $\xi - 1, \ldots \xi - n + 1$ lie outside of the interval ab, some of the conditions (32) need not be imposed. They may, however, in this case be imposed *without at all affecting the function* $u(x)$.

This is a system of linear equations for determining the z's, whose determinant, since the y's are linearly independent solutions of (28), is not zero. The functions z are therefore uniquely determined and are called the functions *adjoint* to $y_1, \cdots y_n$. It should be noticed that whereas the y's are defined when $a \leqq x \leqq b$, the z's are defined only when $a \leqq \xi \leqq b - n$.

LEMMA. *The functions* $z_1(\xi), \cdots z_n(\xi)$ *are linearly independent.*

If they were not, the determinant

$$\Delta = \begin{vmatrix} z_1(\xi) & \cdots z_n(\xi) \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ z_1(\xi - n + 1) & \cdots z_n(\xi - n + 1) \end{vmatrix}$$

would necessarily vanish for all values of ξ from a to $b - n$ inclusive. This is impossible since, when we form the product of Δ by the determinant

$$\begin{vmatrix} y_1(\xi + 1) & \cdots y_n(\xi + 1) \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ y_1(\xi + n) & \cdots y_n(\xi + n) \end{vmatrix}$$

by combining rows with rows, we get, as we see from (35), a determinant which has zeros everywhere above the secondary diagonal and ones at every point in this diagonal, and which is therefore not zero. This proves our lemma.

Returning now to the function $u(x)$ defined by (31), let us see whether we can make it satisfy the boundary conditions (29). For this purpose it is necessary and sufficient that the quantities c, d satisfy the relations

$$(36)\quad c_1A_i(y_1) + \cdots + c_nA_i(y_n) + d_1B_i(y_1) + \cdots + d_nB_i(y_n) = 0 \qquad (i = 1, 2, \cdots n).$$

Substituting here for the c's their values from (34), we find

$$(37)\quad \begin{aligned} &p_0(\xi)[d_1(\xi)W_i(y_1) + \cdots + d_n(\xi)W_i(y_n)] \\ &\qquad = z_1(\xi)A_i(y_1) + \cdots + z_n(\xi)A_i(y_n) \quad (i = 1, 2, \cdots n). \end{aligned}$$

LEMMA. *A necessary and sufficient condition that equations* (37) *be consistent for all values of* ξ *for which* $a \leqq \xi \leqq b - n$ *is that the determinant* D *of Theorem 1 do not vanish.*

The proof of this lemma follows so closely the proof of the lemma preceding Theorem 4 that it is unnecessary to repeat it.

We now obtain at once

THEOREM 10. *A necessary and sufficient condition for the existence of a Green's function of the system, consisting of the expression in* (28) *together with conditions* (29), *where* (29) *are assumed to be linearly independent, is that the system* (28), (29) *be incompatible.*

COROLLARY. *If the system* (28), (29) *is incompatible, there exists only one Green's function.*

7. An Application of Green's Function.—If the Green's function G used in §6 exists, we know that the semi-homogeneous system (24), (29) has one and only one solution. We wish to prove in this section that this solution is given by the formula

$$u(x) = \sum_{\xi=a}^{b-n} G(x, \xi)p(\xi). \tag{38}$$

The function u determined by (38) may be written

$$\begin{aligned} u(x) &= \sum_{\xi=a}^{x-1} [d_1(\xi)y_1(x) + \cdots + d_n(\xi)y_n(x)]p(\xi) \\ &\qquad + \sum_{\xi=x}^{b-n} [c_1(\xi)y_1(x) + \cdots + c_n(\xi)y_n(x)]p(\xi). \end{aligned} \tag{39}$$

Using formulæ (35), and assuming $x \leqq b - n + 1$, we see that we may write:

$$\begin{aligned} u(x+i) &= \sum_{\xi=a}^{x-1} [d_1(\xi)y_1(x+i) + \cdots + d_n(\xi)y_n(x+i)]p(\xi) \\ &+ \sum_{\xi=x}^{b-n} [c_1(\xi)y_1(x+i) + \cdots + c_n(\xi)y_n(x+i)]p(\xi) \quad (i = 1, 2, \cdots n-1). \end{aligned}$$

Finally if $x \leqq b - n$

$$\begin{aligned} u(x+n) &= \sum_{\xi=a}^{x-1} [d_1(\xi)y_1(x+n) + \cdots + d_n(\xi)y_n(x+n)]p(\xi) \\ &\qquad + \sum_{\xi=x}^{b-n} [c_1(\xi)y_1(x+n) + \cdots + c_n(\xi)y_n(x+n)]p(\xi) + \frac{p(x)}{p_0(x)}. \end{aligned}$$

From these equations it follows that, since the y's are solutions of (28), $u(x)$ satisfies (24). We also find from the values of $u(x)$, $u(x+1)$, $\cdots u(x+n-1)$ just obtained

$$A_i(u) = \sum_{\xi=a}^{b-n} [c_1(\xi)A_i(y_1) + \cdots + c_n(\xi)A_i(y_n)]p(\xi),$$

$$B_i(u) = \sum_{\xi=a}^{b-n} [d_1(\xi)B_i(y_1) + \cdots + d_n(\xi)B_i(y_n)]p(\xi).$$

A reference to (36) shows that the sum of these two quantities is zero; hence

THEOREM 11. *If the system consisting of the first member of* (24) *and the conditions* (29) *has a Green's function* $G(x, \xi)$, *the solution of the semi-homogeneous system* (24), (29) *is given by* (38).

Here too we may obtain, as a very special case, the formula for expressing the solution of (24) which vanishes at the points $a, a + 1, \cdots a + n - 1$ in terms of a fundamental system $y_1, \cdots y_n$ of the reduced equation (28). Here all the c's vanish identically by (36), while the d's are then determined by (34) in terms of the z's. Hence, from (39),

$$u(x) = \sum_{\xi=a}^{x-1} [z_1(\xi)y_1(x) + \cdots + z_n(\xi)y_n(x)] \frac{p(\xi)}{p_0(\xi)}$$

is the desired solution of (24),—the *principal solution* at the point a.

It would now be easy to proceed to parallel the developments of §4 by introducing the adjoint difference equation.* Enough has, however, already been said to indicate how completely the whole theory can be carried over from differential to difference equations.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.
June 10, 1911.

---

* Cf. for instance Wallenberg: Sitzungsberichte d. Berl. math. Ges., 7th year (1908), p. 50.